1 Recap

We have the universal smooth hypersurface of degree d in $\mathbb{P}^n$,

$$\begin{array}{c} \mathcal{Y} \subset \mathbb{P}^n \times B \\ \big\downarrow \pi \\ B \subset H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \end{array}$$

where B is the (Zariski open) smooth locus.

Note 1.1. We need to look at an open subset B, because otherwise the projection is not a submersion hence we can't use Ehresmann. The other way to think about this is that the fiber over a regular point is a smooth submanifold.

Take B° an open subset of B parametrizing hypersurfaces without any non-trivial automorphism, and take the quotient by the $\mathrm{GL}(n+1)$ action on $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$. This induces a quotient on $\mathcal{Y}$ as well, and by abuse of notation we can this new family $\mathcal{Y} \to B$ as well. For any $f \in B$, this is a universal family of deformations for the hypersurface $Y_f = \pi^{-1}(f)$.

Note 1.2. The idea of taking B° is probably to avoid dealing with GIT quotient, since we are essentially removing the non-closed orbits.

This family gives us a period map:

$$\mathscr{P} : B \to \Gamma \backslash D$$

Note 1.3. Infinitesimal Torelli says that this map is an immersion, while generic Torelli says that it has degree 1 over the image.

We can look at this map in more details. Pick $f \in B$, and consider the lattice:

$$V = H^{n-1}(Y_f, \mathbb{Z})_{\text{prim}} = \ker\Big(H^{n-1}(Y_f, \mathbb{Z}) \xrightarrow{\iota_*} H^{n+1}(\mathbb{P}^n, \mathbb{Z})\Big)$$

which can be thought of as $\smile H$ (cup product with hyperplane class). Then we can think of D as living inside $\prod_{p=0}^{n-1} \mathrm{Gr}(h^p, V_{\mathbb{C}})$ where $h^p = \dim F^p V_{\mathbb{C}}$. We can locally identify the differential:

$$\begin{aligned} \mathrm{d}\mathscr{P}_f : T_{B,f} &\to \bigoplus_p \mathrm{Hom}\left({F^p V_{\mathbb{C}}}\big/{F^{p+1} V_{\mathbb{C}}}, {F^{p-1} V_{\mathbb{C}}}\big/{F^p V_{\mathbb{C}}} \right) \\ T_{B,f} &\to \bigoplus_p \mathrm{Hom}\big(H^{p,n-1-p}(Y_f, \mathbb{C})_{\text{prim}}, H^{p-1,n-p}(Y_f, \mathbb{C})_{\text{prim}}\big) \\ u &\mapsto \bigoplus_p \overline{\nabla}_{p,f}(-, u) \end{aligned}$$

where $\overline{\nabla}_{p,f} : {F^p V_{\mathbb{C}}}\big/{F^{p+1} V_{\mathbb{C}}} \to {F^{p-1} V_{\mathbb{C}}}\big/{F^p V_{\mathbb{C}}} \otimes \Omega_{B,f}$ which comes from the Gauss-Manin connection $\nabla : \mathscr{V} \to \mathscr{V} \otimes \Omega_B$ where $\mathscr{V} = V_{\mathbb{C}} \otimes \mathscr{O}_B$.

Definition 1.1. Let $S = \bigoplus_k H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k))$ and J_f be the Jacobian ideal generated by partial derivatives $\frac{\partial f}{\partial X_i}$. The quotient ring is denoted $R_f = {}^{S}\!/_{J_f}$.

Hank showed last week that we can identify $\overline{\nabla}_{p,f}$ with the map given by multiplication

$$\overline{\nabla}_{p,f} : R_f^{(n-p)d-n-1} \to \mathrm{Hom}\Big(R_f^d, R_f^{(n-p+1)d-n-1}\Big)$$

which gives (the map in each coordinate is given by multiplication)

$$\mathrm{d}\mathscr{P}_f : R_f^d \to \bigoplus_p \mathrm{Hom}\Big(R_f^{(n-p)d-n-1}, R_f^{(n-p+1)d-n-1}\Big)$$

2 Infinitesimal Torelli

This theorem says that $\mathrm{d}\mathscr{P}_f$ is injective except for cubic surfaces in $\mathbb{P}^3$ (where there is no Hodge theory since $h^{2,0} = 0$) and quadratic hypersurfaces (where the quotient B° by $\mathrm{GL}(n+1)$ is just a point).

Note 2.1. The whole quadric hypersurface business has to do with all smooth quadratic forms being projectively equivalent to $X_0^2 + X_1^2 + ... + X_n^2$ (the other ones have smaller ranks hence not smooth).

Definition 2.1. Let $S = \mathbb{C}[X_0, X_1, ..., X_n]$ and $\{G_i\}_{i=0}^n$ be a sequence of homogeneous polynomials $G_i \in S^{d_i}$ with no common zero. Let $R_G = {}^{S}\!/_{\langle G_0, G_1, ..., G_n\rangle} = {}^{S}\!/_{J_G}$.

Note 2.2. $\mathbb{V}(J_G) = \emptyset$ since no common zero, and thus by weak Hilbert's Nullstellensatz $1 \in J_G$ thus $J_G^k = S^k$ for k large enough (here we are saying they agree for high enough degree, not talking about powers of ideals). Since J_G and S agrees for large degree, $R_G = {}^{S}\!/_{J_G}$ is finite dimensional as a $\mathbb{C}-$vector space hence R_G is Artinian.

Theorem 2.2 (Macaulay)**.** *Let* $N = \left(\sum_{i=0}^n d_i\right) - n - 1$*. We have* $\dim_{\mathbb{C}} R_G^N = 1$*, and for every* $k \in \mathbb{Z}$ *we have a perfect pairing*

$$R_G^k \times R_G^{N-k} \to R_G^N$$

Corollary 2.3. *We have the following:*

1. $R_G^k \neq 0 \Leftrightarrow 0 \leq k \leq N$.
2. *For* $a, b \in \mathbb{Z}$ *with* $b \geq 0$ *and* $a + b \leq N$*, the map given by product*

$$\mu : R_G^a \to \mathrm{Hom}(R_G^b, R_G^{a+b})$$

   is injective.

Note 2.3. R_G^N is the socle of the ring. See this note which gives that a quotient local ring of dimension 0 (Artinian) is gorenstein iff its socle is 1-dimensional. Localize, the maximal ideal looks like $(X_0, ..., X_N)$, then the socle is the biggest submodule of R_G that is annihilated by the maximal ideal hence is R_G^N

Once we have this corollary, $\mathrm{d}\mathscr{P}_f$ is injective iff it's injective on at least one coordinate hence we are done if we can find some p such that $R_f^d \to \mathrm{Hom}\Big(R_f^{(n-p)d-n-1}, R_f^{(n-p+1)d-n-1}\Big)$ is injective. So we just need to find some p such that

$$(n-p)d-n-1 \geq 0, \quad (n-p+1)d-n-1 \leq (d-1)(n+1)-n-1$$

and this is always possible except for the cubic surface and quadric hypersurfaces cases.

Proof of corollary 2.3. For the first part, clearly $R_G^k = 0$ for $k < 0$. Let $k > N$ then $R_G^{N-k} = 0$ so $R_G^k = 0$. Now consider $0 \leq k \leq N$, suppose that $R_G^k = 0$ then $R_G^l = 0$ for all $l \geq k$ since any polynomial of degree l has a factor of degree k. This in turn implies that $N < k$ since $\dim R_G^N = 1$.

For the second part, consider $p(X) \in \ker \mu \subset R_G^a$ then $p(X)q(X) = 0$ for all $q(X) \in R_G^b$. Then for any $r(X) \in R_G^{N-a-b}$, we have $p(X)q(X)r(X) = 0 \in R_G^N$. On the other hand, any $h(X) \in R_G^{N-a}$ can be factored as $q(X)r(X)$ so the map $R_G^{N-a} \xrightarrow{\cdot p(X)} R_G^N$ is zero. The perfect pairing in Macaulay's theorem gives

$$R_G^k \simeq \mathrm{Hom}(R^{N-k}, R_G^N)$$

hence $p(X) = 0$. Thus μ is injective. □

Note 2.4. The proof is essentially correct, but it's very important that R_G is artinian here. In an artinian ring, primes are maximals hence the only irreducibles are linear factors. It's not true that we have factorization in $\mathbb{C}[X_0, ..., X_n]$.

Proof of theorem 2.2. Let $\mathscr{L} = \bigoplus_{i=0}^n \mathscr{O}_{\mathbb{P}^n}(-d_i)$ then we get morphism

$$s : \mathscr{L} \xrightarrow{\begin{pmatrix} G_0 & G_1 & \dots & G_n \end{pmatrix}} \mathscr{O}_{\mathbb{P}^n}$$

then the dual $s^\vee$, given by the transpose of $\begin{pmatrix} G_0 & G_1 & \dots & G_n \end{pmatrix}$, can be thought of as a section of $\mathscr{L}^\vee$. Furthermore, $J_G^k = \mathrm{im}\, s(k) : H^0(\mathbb{P}^n, \mathscr{L}(k)) \to H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k))$. Let $Z = \mathbb{V}(s^\vee)$ then we have the Koszul resolution

$$0 \to \bigwedge^{n+1} \mathscr{L} \to \bigwedge^{n} \mathscr{L} \to ... \to \mathscr{L} \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_Z \to 0$$

Now it's clear, from the matrix form, that the zero locus of $s^\vee$ is $\mathbb{V}(J_G)$ which is empty, hence the complex

$$0 \to \bigwedge^{n+1} \mathscr{L} \to \bigwedge^{n} \mathscr{L} \to ... \to \mathscr{L} \to \mathscr{O}_{\mathbb{P}^n} \to 0$$

is acyclic. Call this complex $0 \to \mathscr{L}^0 \to \mathscr{L}^1 \to ... \to \mathscr{L}^n \to \mathscr{L}^{n+1} \to 0$. Now we have the following spectral sequence of filtered complex $A^\bullet$,

$$E_1^{p,q} = \mathrm{R}^q F(A^p) \Rightarrow \mathrm{R}^{p+q} F(A^\bullet)$$

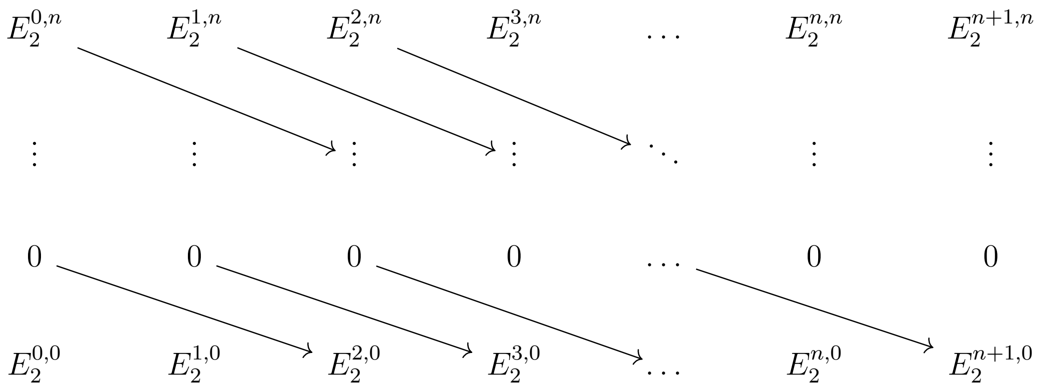
In our case, let $F = \Gamma$ and $A^\bullet = \mathscr{L}^\bullet(k)$ then the sequence becomes

$$E_1^{p,q} = H^q(\mathbb{P}^n, \mathscr{L}^p(k)) \Rightarrow \mathbb{H}^{p+q}(\mathbb{P}^n, \mathscr{L}^\bullet(k))$$

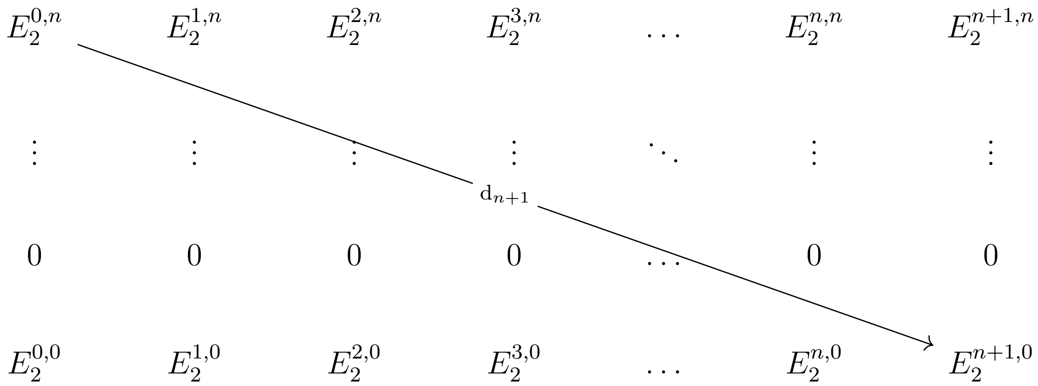
Now $\mathscr{L}^\bullet$ is acyclic hence $\mathscr{L}^\bullet(k)$ is acyclic, hence trivial in the derived category. Thus the hypercohomology is 0 (in general if $A^\bullet$ is acyclic then $\mathrm{R}^i F(A^\bullet) = \mathscr{H}^i(\mathrm{R}F(A^\bullet)) = 0$ by the same reasoning). On the other hand, $\mathscr{L}^q(k)$ is a direct sum of line bundle so by Hartshorne's p. 209 (colimit commutes with cohomology), $E_1^{p,q} = 0$ unless $q = 0, n$.

Note 2.5. $\bigwedge^n(M \oplus N) = \bigoplus_{p+q=n} \bigwedge^p(M) \otimes \bigwedge^q(N)$.

$$\begin{array}{ccccccccccccc}
E_1^{0,n} & \longrightarrow & E_1^{1,n} & \longrightarrow & E_1^{2,n} & \longrightarrow & E_1^{3,n} & \longrightarrow & \dots & \longrightarrow & E_1^{n,n} & \longrightarrow & E_1^{n+1,n} \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 \\
E_1^{0,0} & \longrightarrow & E_1^{1,0} & \longrightarrow & E_1^{2,0} & \longrightarrow & E_1^{3,0} & \longrightarrow & \dots & \longrightarrow & E_1^{n,0} & \longrightarrow & E_1^{n+1,0}
\end{array}$$



hence the E_3 page looks the same, just with different arrows. Now these arrows are still either coming from 0 or pointing to 0 till the E_{n+1} page, i.e., $\mathrm{d}_r : E_r^{p,q} \to E^{p+r,q-r+1}$ is 0 for $2 \le r \le n$ hence $E_2^{\bullet,\bullet} = E_3^{\bullet,\bullet} = ... = E_{n+1}^{\bullet,\bullet}$. The E_{n+1} page looks like



thus the E_∞ page looks like

$$\begin{array}{ccccccc}
\ker \mathrm{d}_{n+1} & E_2^{1,n} & E_2^{2,n} & E_2^{3,n} & \dots & E_2^{n,n} & E_2^{n+1,n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & \dots & E_2^{n,0} & \mathrm{Coker}\ \mathrm{d}_{n+1}
\end{array}$$

and since this converges to hypercohomology which is 0, we have all these terms equal to 0. Thus we have an isomorphism between $E_2^{0,n}$ and $E_2^{n+1,0}$. Now,

$$\begin{aligned}
E_2^{n+1,0} &= \mathrm{Coker}\big(E_1^{n,0} \to E_1^{n+1,0}\big) \\
&= \mathrm{Coker}\Big(H^0(\mathbb{P}^n, \mathscr{L}(k)) \xrightarrow{s} H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k))\Big) = S^k/J_G^k = R_G^k \\
E_2^{0,n} &= \ker\big(E_1^{0,n} \to E_1^{1,n}\big) \\
&= \ker\left(H^n\left(\mathbb{P}^n, \bigwedge^{n+1}\mathscr{L}(k)\right) \xrightarrow{s} H^n\left(\mathbb{P}^n, \bigwedge^{n}\mathscr{L}(k)\right)\right)
\end{aligned}$$

On the other hand, we have

$$\bigwedge^{n+1}\mathscr{L}(k) = \bigotimes_{i=0}^{n}\mathscr{O}_{\mathbb{P}^n}(-d_i) = \mathscr{O}_{\mathbb{P}^n}\left(k - \sum_{i=0}^{n} d_i\right)$$

since $\mathscr{L} = \bigoplus_{i=0}^n \mathscr{O}_{\mathbb{P}^n}(-d_i)$, and

$$\begin{aligned}
\bigwedge^{n}\mathscr{L}(k) &= \mathscr{O}_{\mathbb{P}^n}(k) \otimes \bigwedge^{n}\mathscr{L} \\
&= \mathscr{O}_{\mathbb{P}^n}(k) \otimes \mathscr{L}^\vee \otimes \bigwedge^{n+1}\mathscr{L} \\
&= \mathscr{H}om\left(\mathscr{L}, \bigwedge^{n+1}\mathscr{L}(k)\right) \\
&= \mathscr{H}om\left(\mathscr{L}, \mathscr{O}_{\mathbb{P}^n}\left(k - \sum_{i=0}^{n} d_i\right)\right) \\
&= \mathscr{L}^\vee\left(k - \sum_{i=0}^{n} d_i\right)
\end{aligned}$$

Note 2.6. We need to check that $\mathscr{L}^\vee \otimes \bigwedge^{n+1}\mathscr{L} \simeq \bigwedge^n \mathscr{L}$. This is only really true in this case because $\wedge^{n+1}\mathscr{L}$ is a line bundle. In the normal case, suppose $\mathscr{L}$ has rank m, then a comparison of dimension gives $m \cdot \binom{m}{n+1} = \binom{m}{n}$ which has little chance of being true.

By Serre duality, we get

$$\begin{aligned}
\left(E_2^{0,n}\right)^\vee &= \mathrm{Coker}\left(H^0\left(\mathbb{P}^n, \mathscr{L}\left(-n-1-k+\sum_{i=0}^{n} d_i\right)\right) \to H^0\left(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}\left(-n-1-k+\sum_{i=0}^{n} d_i\right)\right)\right) \\
&= R_G^{-n-1-k+\sum d_i} = R_G^{N-k}
\end{aligned}$$

thus we get an isomorphism

$$\mathrm{d}_{n+1} : \left(R_G^{N-k}\right)^\vee \to R_G^k$$

To conclude the perfect pairing we just need to check that the isomorphisms is compatible with multiplications, i.e., the following diagram is commutative

$$\begin{array}{ccc} \left(R_G^{N-k}\right)^\vee & \xrightarrow{\mathrm{d}_{n+1}} & R_G^k \\ \Big\downarrow{\scriptstyle (\cdot p)^\vee} & & \Big\downarrow{\scriptstyle \cdot p} \\ \left(R_G^{N-k-l}\right)^\vee & \xrightarrow[\mathrm{d}_{n+1}]{} & R_G^{k+l} \end{array}$$

which should follow from the fact that our section s was defined using multiplications. □

3 Generic Torelli

For generic Torelli, we need the symmetriser lemma

Lemma 3.1. *Let*

$$T^{a,b} = \left\{ \phi \in \mathrm{Hom}(R_G^a, R_G^b) \middle| p(X) \cdot \phi(q(X)) = \phi(p(X)) \cdot q(X) \ \forall \ p(X), q(X) \in R_G^a \right\}$$

If $a + b < N$ and $\max_i(d_i + b) \leq N$ then we have

$$\mu(R_G^{b-a}) = T^{a,b} \subset \mathrm{Hom}(R_G^a, R_G^b)$$

Theorem 3.2 (Generic Torelli)**.** *The period map $\mathscr{P} : B \to \Gamma \backslash D$ has degree 1 over its image, with the following possible exceptions:*

1. *d divides $n + 1$;*
2. *$d = 3, n = 3$, i.e., cubic surfaces in $\mathbb{P}^2$;*
3. *$d = 4, n \equiv 1 \mod 4$;*
4. *$d = 6, n \equiv 2 \mod 6$.*

Note 3.1. The statement for quadric hypersurfaces is trivial, since B is just a single point.

Voisin's argument (in her book) on $(H^{n-1}(Y_f, \mathbb{Z})_{\mathrm{prim}}, F^\bullet) \simeq (H^{n-1}(Y_g, \mathbb{Z})_{\mathrm{prim}}, F^\bullet)$, with very general f, inducing an isomorphism of variations of Hodge structures on neighborhoods $U \ni f$ and $V \ni g$ is sketchy. See her 2020 paper on extending generic Torelli to see a (seemingly) clearer argument.

Note 3.2. The argument in Voisin's book is correct. The idea is that the period map is an immersion, hence locally (on the target) it looks like a covering map. If 2 points $f, g \in B$ get mapped the same Hodge structure then we have 2 neighborhoods $U \ni f, V \ni g$ mapping isomorphically to the same neighborhood of the Hodge structure in D.

The moral of the story is that such an isomorphism induces a commutative diagram

$$\begin{array}{ccc} R_f^d & \xrightarrow{\mathrm{d}\mathscr{P}_f} & \bigoplus_p \operatorname{Hom}\Big(R_f^{(n-p)d-n-1}, R_f^{(n-p+1)d-n-1}\Big) \\ \Big\downarrow \simeq & & \Big\downarrow \simeq \\ R_g^d & \xrightarrow[\mathrm{d}\mathscr{P}_g]{} & \bigoplus_p \operatorname{Hom}\Big(R_g^{(n-p)d-n-1}, R_g^{(n-p+1)d-n-1}\Big) \end{array}$$

and we claim that such a diagram is enough to conclude Y_f and Y_g are isomorphic. Let k be the smallest non-zero integer that can be written as $k = (n-p)d - n - 1$. Since d does not divide $n+1$, $k < d$ (since we can change the RHS by $\pm d$). Then we have a diagram

$$\begin{array}{ccc} R_f^d & \xrightarrow{\mu} & \operatorname{Hom}\big(R_f^k, R_f^{k+d}\big) \\ \Big\downarrow \iota_d & & \Big\downarrow \varphi \mapsto \iota_{k+d}\circ\varphi\circ\iota_k^{-1} \\ R_g^d & \xrightarrow[\mu]{} & \operatorname{Hom}\big(R_g^k, R_g^{k+d}\big) \end{array}$$

By the symmetriser theorem, we can identify the image of R_f^{d-k} under multiplication with

$$T_f^{k,d} = \Big\{\phi \in \operatorname{Hom}(R_f^k, R_f^d) \Big| p(X)\cdot\phi(q(X)) = \phi(p(X))\cdot q(X)\ \forall\ p(X), q(X) \in R_f^k\Big\}$$

Note that we have $R_f^k \simeq R_g^k$ through ι_k (since these are the same graded pieces of isomorphic Hodge structures) and similarly with ι_{k+d}. Consider $\alpha(X) \in R_f^{d-k}$, define a map $\phi \in \operatorname{Hom}(R_g^k, R_g^d)$ as follows: for any $A_g(X) \in R_g^k$ there is $A_f(X) \in R_f^k$ such that $\iota_k(A_f(X)) = A_g(X)$, and we define $\phi(A_g(X)) = \iota_d(\alpha(X)\cdot A_f(X))$. In other words,

$$\phi(A_g(X)) = \iota_d(\alpha(X)\cdot\iota_k^{-1}(A_g(X)))$$

which gives the $\mathbb{C}-$linear structure of ϕ for free. Now let $B_g(X) \in R_g^k$, then the above diagram gives

$$\begin{aligned} B_g(X)\cdot\phi(A_g(X)) &= B_g(X)\cdot\iota_d(\alpha(X)\cdot\iota_k^{-1}(A_g(X))) \\ &= (\mu\circ\iota_d)(\alpha(X)\cdot\iota_k^{-1}(A_g(X)))(B_g(X)) \\ &= ((\varphi\mapsto\iota_{k+d}\circ\varphi\circ\iota_k^{-1})\circ\mu)(\alpha(X)\cdot\iota_k^{-1}(A_g(X)))(B_g(X)) \\ &= (\varphi\mapsto\iota_{k+d}\circ\varphi\circ\iota_k^{-1})\big(\bullet\mapsto(\bullet)\cdot\alpha(X)\cdot\iota_k^{-1}(A_g(X))\big)(B_g(X)) \\ &= \iota_{k+d}\circ\big(\bullet\mapsto(\bullet)\cdot\alpha(X)\cdot\iota_k^{-1}(A_g(X))\big)\circ\iota_k^{-1}(B_g(X)) \\ &= \iota_{k+d}\big(\iota_k^{-1}(B_g(X))\cdot\alpha(X)\cdot\iota_k^{-1}(A_g(X))\big) \\ &= \iota_{k+d}\big(\iota_k^{-1}(A_g(X))\cdot\alpha(X)\cdot\iota_k^{-1}(B_g(X))\big) \\ &= \iota_{k+d}\circ\big(\bullet\mapsto(\bullet)\cdot\alpha(X)\cdot\iota_k^{-1}(B_g(X))\big)\circ\iota_k^{-1}(A_g(X)) \\ &= A_g(X)\cdot\phi(B_g(X)) \end{aligned}$$

so by symmetriser lemma, $\phi = \mu(\beta(X))$ for some $\beta(X) \in R_g^k$. To check that this correspondence $\alpha(X) \mapsto \beta(X)$ is $\mathbb{C}-$linear we probably just need to show that these $T^{k,d}$ are

$\mathbb{C}-$subspace and $\mu^{-1} : T^{k,d} \to R^{d-k}$ is $\mathbb{C}-$linear. In summary we get a a new isomorphism $\iota_{d-k} : R_f^{d-k} \to R_g^{d-k}$, with a new diagram

$$\begin{array}{ccc} R_f^{d-k} & \xrightarrow{\ \mu\ } & \mathrm{Hom}\big(R_f^k, R_f^d\big) \\ \Big\downarrow{\scriptstyle \iota_{d-k}} & & \Big\downarrow{\scriptstyle \varphi \mapsto \iota_d \circ \varphi \circ \iota_k^{-1}} \\ R_g^{d-k} & \xrightarrow[\ \mu\ ]{} & \mathrm{Hom}\big(R_g^k, R_g^d\big) \end{array}$$

Iterating this process, for $\delta = \gcd(d, n+1)$ we get an isomorphism

$$R_f^{(\delta)} \simeq R_g^{(\delta)}$$

which are subrings consisting of degrees divisible by δ. The claim is that for $\delta < d$ we can recover the ring structure on R_f, R_g which gives $J_f \simeq J_g$. Mather-Yau's theorem then says that Y_f and Y_g are projectively equivalent, and we are done.

Note 3.3. The idea seems to be that for $\delta < d$, $R_f^\delta \simeq S^\delta \simeq R_g^\delta$. This implies

$$S^d \simeq \mathrm{Sym}^{d/\delta} S^\delta \simeq \mathrm{Sym}^{d/\delta} R_f^\delta \to R_f^d$$

is surjective with kernel J_f. The map $\mathrm{Sym}^{d/\delta} S^\delta \to R_g^d$ with kernel J_g is the same map since we have identified both R_f^δ, R_g^δ with S^δ and this identification respects multiplication. It follows that $J_f \simeq J_g$.